

## ON SOME ANALYTICAL PROPERTIES OF STABLE DENSITIES

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ABSTRACT. L. Bondesson [1] conjectured that the density of a positive  $\alpha$ -stable distribution is hyperbolically completely monotone (HCM in short) if and only if  $\alpha \leq 1/2$ . This was proved recently by P. Bosch and Th. Simon, who also conjectured a strengthened version of this result. We disprove this conjecture as well as a correlated conjecture of Bondesson, while giving a short new proof of the initial conjecture, as a direct consequence of a new algebraic property of HCM and Generalized Gamma convolution densities (GGC in short) which we establish.

**Résumé**

L. Bondesson a conjecturé que la densité d'une variable aléatoire  $\alpha$ -stable positive est hyperboliquement complètement monotone (HCM) si et seulement si  $\alpha \leq 1/2$ . Ce résultat a été récemment établi par P. Bosch et Th. Simon qui ont aussi conjecturé une version plus forte de ce résultat. Nous infirmons celle-ci ainsi qu'une autre conjecture de L. Bondesson. Nous donnons aussi une courte et nouvelle preuve de la conjecture initiale, comme conséquence directe d'une nouvelle propriété algébrique des fonctions HCM et des densités gamma généralisées (GGC) que nous établissons.

## 1. INTRODUCTION

This paper is concerned with the HCM property for stable distributions and GGC random variables, whose definitions we recall below. Hyperbolically completely monotone functions (HCM in short) were introduced by Lennart Bondesson [1] in order to analyze infinitely divisible distributions. On the other hand, the generalized gamma convolutions (GGC in short) introduced by O. Thorin [7], are the weak limits of finite convolutions of Gamma random variables. These notions are closely related, indeed the main example of HCM functions are the Laplace transform of GGC variables. L. Bondesson proved, in [1], that the  $\alpha$ -stable positive random variables (denoted  $S_\alpha$ ), with density  $g_\alpha$ , are GGC for all  $\alpha \in ]0, 1]$  and that they have an HCM-density when  $\alpha = n^{-1}$ , for any integer  $n \geq 2$ . He also conjectured that this property holds for all  $\alpha \leq 1/2$ . In a previous preprint [4] we proved that the density (denoted  $G_\alpha$ ) of  $S_\alpha^{-\beta}$ , (with  $\beta := \frac{\alpha}{1-\alpha}$ ) is HCM if  $\alpha \in [1/3, 1/2]$ . This implies easily the HCM property of  $g_\alpha$  for  $\alpha$  in this range. Moreover, it is easy to see that  $\beta$  is the largest real number for which this property holds. Recently, Pierre Bosch and Thomas Simon [3] proved the full original Bondesson conjecture. Their proof makes use of the following result from Bondesson [2] "The independent product or ratio of two GGC random variables is again GGC". Furthermore they conjectured that  $G_\alpha$  is also an HCM function for all  $\alpha \leq 1/2$ . In the present paper, we prove that actually  $G_\alpha$  is not HCM for  $\alpha < 1/3$ . Moreover, for  $\alpha \in ]1/2, 1[$ , using the fact that  $\frac{e^{\delta x}}{G_\alpha(x)}$  is HCM (see also [4]), we obtain that  $G_\alpha$  is not the density of a GGC random variable. Since  $g_\alpha$  is a GGC-density, this gives an example of a GGC

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random variable  $S_\alpha$  such that  $S_\alpha^\gamma$  is not GGC and  $|\gamma| > 1$ , thus providing a negative answer to a question of L. Bondesson [2]. Finally, using Bondesson new remarkable property already mentioned, we prove that the multiplicative convolution of an HCM function and a GGC density is again HCM. As we show, initial Bondesson's conjecture is an immediate consequence of this result.

The central result of this paper is a representation of  $G_\alpha$  for all  $\alpha \in ]0, 1]$ . One consequence of this representation is the estimate of this density by a convex combination of two gamma densities, namely  $\Gamma(1/2, \delta)$  and  $\Gamma(\alpha, \delta)$ , with  $\delta = (1 - \alpha)\alpha^{\frac{\alpha}{1-\alpha}}$ . Hopefully, this might be useful for the numerical investigation of these functions.

This paper is organized as follow : In the first part, we recall the facts on HCM functions and GGC random variables which are used in the sequel, we refer to Bondesson [1] and James-Roynette-Yor [5] for more general informations on the subject, other examples of GGC or HCM functions can be found in the recent work of W. Jedidi and Th. Simon [6]. In the second part, we do the same for  $\alpha$ -stable densities, for which we refer to Zolotarev [8].

In part 3, we give a first rough estimate of  $G_\alpha$ , as a function of a complex variable, obtained using the saddle point method. In part 4, the main result is given : we give a representation of the density  $G_\alpha$  of  $S_\alpha^{-\beta}$  for every  $\alpha \in ]0, 1[$  and, as a first corollary, an estimate of  $S_\alpha^{-\beta}$  by a convex combination of two Gamma distributions is given. In part 5, we prove that  $G_\alpha$  is not HCM for  $\alpha < 1/3$  (which disproves the Simon-Bosch conjecture), is HCM for  $\alpha \in [1/3, 1/2]$  and anti-HCM (see definition below) for  $\alpha > 1/2$ . As consequences, more precise estimates are given when  $\alpha \in [1/3, 1/2]$  and a corollary of the anti-HCM property for  $\alpha > 1/2$  is that  $S_\alpha$  is GGC although  $S_\alpha^{-\beta}$  is not. In part 6, we prove that the convolution product (sometimes called mixing) of an HCM function and a GGC density is again HCM. As a consequence, we obtain a short proof of the HCM property of  $S_\alpha$ .

## 2. PRELIMINARIES

**2.1. Hyperbolically completely monotone functions.** We recall here the basic definition and properties of the class of hyperbolically completely monotone functions, and refer to [1] for more details.

**Definition 2.1.** *A real positive valued function  $H$  defined on  $]0, +\infty[$  is called hyperbolically completely monotone (HCM) if, for every  $u > 0$  the function  $H(uv)H(uv^{-1})$  is a completely monotone function of the variable  $v + v^{-1}$ .*

Bondesson [1] has obtained the following characterization of HCM functions.

**Proposition 2.2.**

*$H$  is HCM if and only if it admits the following representation*

$$(2.1) \quad H(x) = cx^{\beta-1} \exp \left( -ax - \int_1^\infty \log \frac{x+t}{1+t} \mu_1(dt) - bx^{-1} - \int_1^\infty \log \frac{x^{-1}+t}{1+t} \mu_2(dt) \right)$$

where  $a, b, c$  are non negative constants and  $\mu_1$  and  $\mu_2$  are positive Radon measures on  $[1, +\infty[$  that integrate  $1/t$  at infinity.

We shall use a slightly different but equivalent representation of HCM functions, obtained by an integration by part from (2.2). Denote

$$\theta(t) = \mu_1([1, t])1_{t \geq 1} - \mu_2([1, 1/t])1_{t < 1} + (\beta - 1)$$

then  $\theta$  is a (signed) non decreasing càdlàg function. The following is an immediate consequence of Proposition 2.2.

**Corollary 2.3.**  *$H$  is HCM if and only if it admits the following representation,*

$$(2.2) \quad H(x) = c \exp(-ax - bx^{-1}) \exp \left( \int_0^\infty \left( \frac{1}{x+t} - \frac{1}{t+1} \right) \theta(t) dt \right)$$

where  $a, b, c$  are non negative constants and  $\theta$  is a signed non decreasing function such that  $\int_0^\infty (1 \wedge \frac{1}{t^2}) |\theta(t)| dt < +\infty$

One has  $H(1) = ce^{-a-b}$ . Moreover, if  $\theta = \theta_0$  is a constant function and if  $a = b = 0$ , then  $H(x) = cx^{-\theta_0}$ . The integral condition  $\int_0^\infty (1 \wedge \frac{1}{t^2}) |\theta(t)| dt < +\infty$  is the minimal condition to ensure finite values for  $H(x)$  for every  $x > 0$ . Note also that  $H(x)$  may be infinite at  $x = 0$  and  $x = +\infty$ . In the sequel, the functions admitting this representation with  $\theta$  a non increasing function instead of a non decreasing function will be called **anti-HCM functions**.

The representation (2.2) implies that  $H$  has an analytic continuation on  $\mathbf{C} \setminus ]-\infty, 0]$ . If we denote this continuation by  $H$  again one has, using well known properties of the Stieltjes-Cauchy transform:

$$H(-r^+) := \lim_{z \rightarrow -r, \Im(z) > 0} H(z) = R(r) e^{-i\pi\theta(r)}$$

where  $R(r) e^{-i\pi\theta(r)}$  is the polar decomposition of the complex number  $H(-r^+)$ . This property will play a crucial role in the sequel.

**Definition 2.4.** *The generalized Gamma convolutions (GGC) are the random variables which belong to the smallest class containing Gamma distributions and closed under taking sums of independent variables and weak limits.*

The following results can be found in Bondesson [1]

**Proposition 2.5.**

- (1) *A random variable is GGC if and only if its Laplace transform is an HCM function.*
- (2) *An HCM function  $H$  is the Laplace transform of a random variable if and only  $b = 0, H(0) = 1$ , and the function  $\theta$  in the representation (2.2) is non negative. Moreover, when these properties are satisfied,  $H$  is the Laplace transform of a GGC random variable.*

The class of HCM functions and GGC random variables, have been much studied. We refer mainly to Bondesson monography [1] and to Yor-Roynette-James [5], for GGC- random variables. Note that  $e^{-x^\alpha}$ ,  $\alpha < 1$ , is an HCM function with  $a = b = 0$  and  $\theta(t) = \sin \pi \alpha t^\alpha$ , while  $e^{-x^\alpha}$  is the Laplace transform of the positive  $\alpha$ -stable distribution  $S_\alpha$ , thus  $S_\alpha$  is GGC.

**2.2. Stable random variables.** Let  $\alpha \in ]0, 2[, \rho \in ]0, 1]$ , and suppose  $\gamma = \alpha\rho \in ]0, 1]$  and let  $g_{\alpha,\gamma}(x)$  denote the density of the normalized  $\alpha$ -stable random variable  $S_{\alpha,\rho}$  with asymmetry parameter  $\rho$  ( $\rho = \mathbf{P}(S_{(\alpha,\rho)} > 0)$ ) cf [8]). For  $\rho = 1$  and  $\gamma = \alpha\rho = \alpha \in ]0, 1[$  (and only for these values) this distribution is supported on the half axis  $]0, +\infty[$  and we simply put  $g_\alpha = g_{\alpha,\alpha}$ .

The function  $g_{\alpha,\gamma}$  has Fourier transform

$$e^{-(iu)^\gamma (-iu)^{\alpha-\gamma}} = \int_{-\infty}^{+\infty} e^{-iut} g_{\alpha,\gamma}(t) dt$$

where  $t^\alpha = \exp(\alpha \log(t))$  with  $\log$  the principal determination of the logarithm.

The following integral representation (cf Zolotarev [8]) can be easily obtained by Fourier inversion.

**Lemma 2.6. Zolotarev** For  $r > 0, 0 \leq \alpha < 1$  and  $0 \leq \gamma \leq 1$

$$(2.3) \quad g_{\alpha,\gamma}(r) = (2i\pi)^{-1} \int_0^\infty (e^{-rt-t^\alpha e^{i\pi\gamma}} - e^{-rt-t^\alpha e^{-i\pi\gamma}}) dt$$

The above integral is well defined for all  $\alpha \in ]0, 1[$  and  $|\gamma| \leq 1$ . We will use it as a definition in these cases.

**Lemma 2.7.** The function  $\tilde{g}_{\alpha,\gamma}(x) = x^{-1-\alpha} g_{\alpha,\gamma}(x^{-1})$  is

- (1) decreasing on  $]0, +\infty[$  if  $0 \leq \gamma \leq \alpha \leq 1$ .
- (2) completely monotone if  $0 \leq \gamma \leq \alpha \leq 1/2$

*Proof.* Recall that, if  $X$  is a stable variable with parameters  $(2\alpha, \rho)$  and  $Y$  is an independent stable variable with parameters  $(1/2, 1)$ , then  $Z = XY^{\frac{1}{2\alpha}}$  is a stable variable with parameters  $(\alpha, \rho)$ . Since  $g_{1/2}(t) = \frac{e^{-\frac{1}{2t}}}{\sqrt{2\pi t^3}}$  one has

$$g_{\alpha,\gamma}(x) = 2\alpha \int_0^\infty g_{2\alpha,\gamma}(y) \frac{e^{-\frac{1}{2}(y/x)^{2\alpha}} y^\alpha}{\sqrt{2\pi} x^{\alpha+1}} dy$$

Therefore

$$x^{-1-\alpha} g_{\alpha,\rho}(x^{-1}) = 2\alpha \int_0^\infty g_{2\alpha,\rho}(y) \frac{e^{-\frac{1}{2}(yx)^{2\alpha}} y^\alpha}{\sqrt{2\pi}} dy$$

which is decreasing in  $x$  and completely monotone if  $2\alpha \leq 1$ . □

**Lemma 2.8.** For  $\alpha \leq \delta < 1$

$$\int_0^\infty g_{\alpha,\gamma}(xy) g_\delta(y) y dy = x^{\delta-1} g_{\frac{\alpha}{\delta},\gamma}(x^\delta)$$

*Proof.* Let  $\bar{g}$  denote the tail function of  $g$ ,

$$\bar{g}(x) = \int_x^\infty g(y) dy.$$

Instead of the identity of the lemma, we rather prove the equivalent identity on the associated tail functions,

$$\int_0^\infty \bar{g}_{\alpha,\gamma}(xy) g_\delta(y) dy = \frac{1}{\delta} \bar{g}_{\frac{\alpha}{\delta},\gamma}(x^\delta)$$

By (2.3)

$$\bar{g}_{\alpha,\gamma}(r) = (2i\pi)^{-1} \int_0^\infty (e^{-rt-t^\alpha e^{i\pi\gamma}} - e^{-rt-t^\alpha e^{-i\pi\gamma}}) \frac{dt}{t}$$

therefore

$$\int_0^{+\infty} \bar{g}_{\alpha,\gamma}(xy) g_\delta(y) dy = (2i\pi)^{-1} \int_0^\infty (e^{-x^\delta t^\delta - t^\alpha e^{i\pi\gamma}} - e^{-x^\delta t^\delta - t^\alpha e^{-i\pi\gamma}}) \frac{dt}{t}$$

The proof follows by a simple change of variable ( $t \rightarrow t^{1/\delta}$ ) in the integral and (2.3) again.  $\square$

**Proposition 2.9.** *If  $\alpha < \gamma \wedge 1/2 < 1$  then  $g_{\alpha,\gamma}(x)$  is not of constant sign.*

*Proof.* If  $\gamma \leq 1/2$ , let  $\delta = \frac{\alpha}{\gamma} < 1$  then the complete monotonicity of  $y^{-1-\delta}g_\delta(y^{-1})$  and the positivity of  $g_{\alpha,\gamma}(x)$  would imply the complete monotonicity of  $g_{\gamma,\gamma}(x^\gamma)$ , but this cannot be true since  $g_{\gamma,\gamma}(x^\gamma)$  is not monotonous.

If  $\gamma > 1/2$  and  $\alpha \leq 1/2$  then take  $\delta = \alpha$  and obtain that  $g_{1,\gamma}(x^\alpha)$  would be completely monotonous and this is not true. Actually this function is not monotonous.  $\square$

**Remark** Iterating the convolution with  $\delta = 1/2$  sufficiently, we could obtain that  $g_{\alpha,\gamma}(x)$  is not of constant sign, for all  $\alpha < \gamma \leq 1$ .

Let  $\alpha \in ]0, 1[$ ,  $\beta := \frac{\alpha}{1-\alpha}$  and, for all  $x \in ]0, +\infty[$ ,

$$G_\alpha(x) := \beta^{-1}x^{-\frac{1}{\alpha}}g_\alpha(x^{-\beta^{-1}})$$

The function  $G_\alpha$  is the density of the distribution of  $S_\alpha^{-\beta}$ . It will play an important role in this paper. The following integral representation

$$(2.4) \quad G_\alpha(x) = (2i\pi\beta x)^{-1} \int_0^\infty (e^{-t-e^{i\pi\alpha}t^\alpha x^{1-\alpha}} - e^{-t-e^{-i\pi\alpha}t^\alpha x^{1-\alpha}})dt \quad x > 0$$

shows that  $G_\alpha$  has an analytic continuation to  $\mathbf{C} \setminus ]-\infty, 0]$ , still denoted  $G_\alpha(z)$ .

### 3. A ROUGH ESTIMATE OF $G_\alpha$

Let  $t_0$  be the minimum of the function  $f(t) = t - t^\alpha$  for  $t \in ]0, +\infty[$ , and  $\delta = -f(t_0)$ , i.e.  $t_0 = \alpha^{\frac{1}{1-\alpha}}$  and  $\delta = (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}$ . Define  $f_0(t) = f(t) - f(t_0)$ . The next lemma gives a technical intermediate result that will be improved in the next section.

**Lemma 3.1.** *There exist constants  $A > 0$  and  $B > 0$  such that for all  $z \in \mathbf{C} \setminus ]-\infty, 0]$*

$$|G_\alpha(z)e^{\delta z}| \leq A + B|z|^{-1}$$

Before proving this Lemma we need a new representation of  $G_\alpha$ . Consider the analytic function on  $\mathbf{C} \setminus [0, +\infty[$  which coincides with the principal determination  $t^\alpha$  on the upper half plane. Let  $f_+(t)$  be the function obtained from  $f_0$  by replacing  $t^\alpha$  by this function. In other words,

$$f_+(t) = f_0(t) = t - t^\alpha + \delta \quad \text{if} \quad \Im(t) > 0$$

$$f_+(t) = t - e^{2i\pi\alpha}t^\alpha + \delta \quad \text{if} \quad \Im(t) < 0.$$

Similarly, let

$$f_-(t) = t - e^{-2i\pi\alpha}t^\alpha + \delta \quad \text{if} \quad \Im(t) > 0$$

$$f_-(t) = f_0(t) = t - t^\alpha + \delta \quad \text{if} \quad \Im(t) < 0.$$

One has

$$f_+(\bar{z}) = \overline{f_-(z)}$$

**Lemma 3.2.** *Let  $\theta \in [-1, -(1/2 - \alpha)^+ \cup (1/2 - \alpha)^+, 1]$ , there exist two continuous complex valued functions,  $v_\theta^+(r)$ ,  $v_\theta^-(r)$ , defined for  $r \in [0, +\infty[$ , such that  $v_\theta^+(0) = v_\theta^-(0) = t_0$  and for all  $r > 0$ ,  $v_\theta^+(r) \neq v_\theta^-(r)$  and*

$$f_+(v_\theta^+(r)) = f_-(v_\theta^-(r)) = re^{i\pi\theta}$$

*Proof.* Fix  $\theta \in ]0, 1]$ , we shall build  $v_\theta^+(r)$  and  $v_\theta^-(r)$ . The point  $t_0$  is a non degenerate saddle point for  $f_0$  and  $f_0(t_0) = 0$ , and  $f_0(t) \sim \frac{\alpha(1-\alpha)}{2}(t-t_0)^2$  in a neighborhood of  $t_0$ . By the implicit function theorem there exists two distinct solutions  $z$  satisfying the equation

$$f_0(z) = re^{i\pi\theta}$$

for  $r$  small enough, moreover, one of the two solutions,  $v_\theta^+(r)$  is in the upper half plane and the other one,  $v_\theta^-(r)$ , is in the lower half plane. Since  $f_+ = f_0$  on the upper half plane and  $f_- = f_0$  on the lower half plane, the complex numbers  $v_\theta^+(r)$  and  $v_\theta^-(r)$  satisfy the equations

$$f_0(v_\theta^+(r)) = f_+(v_\theta^+(r)) = re^{i\pi\theta} \quad f_0(v_\theta^-(r)) = f_-(v_\theta^-(r)) = re^{i\pi\theta}$$

Let  $\theta \in ](1/2 - \alpha)^+, 1]$ . We prove that the two functions  $v_\theta^+(r)$  and  $v_\theta^-(r)$  can be continued for all  $r > 0$  and  $\theta \in ](1/2 - \alpha)^+, 1]$ . The case  $-\theta \in ](1/2 - \alpha)^+, 1]$  is obtained by conjugation. Since  $f_+$  is analytic on the upper half plane, by the open mapping theorem, the function  $v_\theta^+$  can be continued as long as  $v_\theta^+(r)$  does not meet the real line. The boundary values of the function  $f_0$  on the upper half plane have imaginary part 0 on the positive real line and negative imaginary part on the negative real line, therefore  $v_\theta^+(r)$  cannot converge to a real point since  $\Im(f_+(v_\theta^+(r))) = r \sin \pi\theta > 0$ .

Similarly  $v_\theta^-$  can be continued as long as  $v_\theta^-(r)$  does not reach the cut  $[0, +\infty[$ . Let  $t \in \mathbf{R}^+$ , the boundary values of  $f_-$  at  $t$  are

$$\begin{aligned} f_-(t^-) &= f_0(t) = t - t^\alpha + \delta \in \mathbf{R}^+ \\ f_-(t^+) &= t - t^\alpha e^{-2i\pi\alpha} + \delta = (t - t^\alpha + \delta) + (1 - e^{-2i\pi\alpha})t^\alpha \\ &= (t - t^\alpha + \delta) + 2t^\alpha \sin \pi\alpha e^{i\pi(1/2-\alpha)} \end{aligned}$$

Since  $t - t^\alpha + \delta \in \mathbf{R}^+$ , we see that  $f_-(t^+)$  and  $f_-(t^-)$  are in the cone  $\{z; |\arg z| \leq |1/2 - \alpha|\}$ . On the other hand  $f_-(v_\theta^-(r)) = re^{i\pi\theta}$  always remains outside this cone, so that  $v_\theta^-(r)$  is defined for all  $r > 0$ .  $\square$

**Lemma 3.3.**

(1) *There exists a positive constant  $A$  such that for  $r \in \mathbf{R}_+$  and  $\theta \notin [-(1/2 - \alpha)^+, (1/2 - \alpha)^+]$ ,*

$$|v_\theta^+(r)| \leq A + 2r \quad |v_\theta^-(r)| \leq A + 2r$$

(2)  *$v_\theta^+(r) \sim re^{i\pi\theta}$  and  $v_\theta^-(r) \sim re^{i\pi\theta}$  for  $r \rightarrow +\infty$  and  $|\theta| > (1/2 - \alpha)^+$ .*

*Proof.* This follows easily from the fact that there exists a constant  $C > 0$  such that  $|\frac{f_\pm(z)}{z} - 1| \leq C|z|^{\alpha-1}$ .  $\square$

**Lemma 3.4.** *For  $\theta \in^c [-(1/2 - \alpha)^+, (1/2 - \alpha)^-]$  and  $\Re(ze^{i\pi\theta}) < 0$ ,*

$$G_\alpha(z) = (2i\pi\beta)^{-1} e^{-\delta z} z e^{i\pi\theta} \cdot \int_0^\infty e^{zte^{i\pi\theta}} (v_\theta^+(t) - v_\theta^-(t)) dt$$

*Proof.* For all  $u < 0$  and  $z \notin ]-\infty, 0]$  one has

$$\begin{aligned} u - u^\alpha e^{i\pi\alpha} z^{1-\alpha} + \delta z &= z f_+\left(\frac{u}{z}\right) \\ u - u^\alpha e^{-i\pi\alpha} z^{1-\alpha} + \delta &= z f_-\left(\frac{u}{z}\right) \end{aligned}$$

Using this, we obtain from (2.4)

$$G_\alpha(z) = (2i\pi\beta)^{-1} \left( \int_{1/2D_{\hat{h}}} e^{zf_+(t)} - \int_{1/2D_{\hat{h}}} e^{zf_-(t)} dt \right)$$

where  $1/2D_{\hat{h}}$  is the half line  $\{-te^{i\pi h}; t \in [0, \infty[ \}$ ,  $h$  is the argument of  $z$ . We change again the contour and replace the half line  $1/2D_{\hat{h}}$  by the curve  $[0, t_0]^+ \cup \{v_h^+(s), s = 0 \rightarrow +\infty[ \}$  for the first integral and  $t \rightarrow [0, t_0]^- \cup \{v_h(s); ]-\infty, s[ \}$  in the second one. Notice also that  $f_+(t^+) = f_-(t^-) = f_0(t)$  for all  $t \in [0, t_0]$ , consequently, the contribution of the two integrals over  $[0, t_0]$  compensate each other and the end point of the half line and the two curves coincide at infinity. Finally we obtain by the use of Cauchy theory that

$$\begin{aligned} G_\alpha(z) &= (2i\pi\beta)^{-1} \left( \int_0^{+\infty} e^{zf_+(v_\theta^+(s))} dv_\theta^+(s) - \int_0^{+\infty} e^{zf_-(v_\theta^-(s))} dv_\theta^-(s) \right) \\ &= (2i\pi\beta)^{-1} \left( \int_0^{+\infty} e^{zse^{i\pi\theta}} (dv_\theta(s) - dv_\theta(s)) \right) \end{aligned}$$

The integral representation of  $G_\alpha$  follows after an integration by part.  $\square$

*Proof of Lemma 3.1.*

If  $z \notin [0, +\infty[$  let  $h$  be such that  $ze^{i\pi h} = -|z|e^{i\pi\varepsilon}$  with  $|\varepsilon| \leq |1/2 - \alpha|$ . One has

$$e^{\delta z} G_\alpha(z) = (2i\pi\beta)^{-1} z \cdot \int_0^{+\infty} e^{-|z|t \cdot e^{i\pi h}} (v_\theta^+(t) - v_\theta^-(t)) dt$$

Using the estimate of  $v_\theta^+(t)$  and  $v_\theta^-(t)$  given in lemma 3.3 we obtain,

$$|e^{\delta z} G_\alpha(z)| \leq (2\pi\beta)^{-1} (A(\sin \pi\alpha)^{-1} + 2(\sin \pi\alpha)^{-2} |z|^{-1})$$

$\square$

#### 4. THE MAIN RESULT

**Theorem 4.1.** *There exists a continuous function  $\theta$ , taking values in  $]0, 1[$ , such that, for all  $z \in \mathbf{C} \setminus ]-\infty, 0]$ ,  $\alpha \in ]0, 1[$ ,*

$$(4.1) \quad G_\alpha(z) = G_\alpha(1) e^{-\delta(z-1)} \exp \int_0^\infty \left( \frac{1}{z+t} - \frac{1}{t+1} \right) \theta(t) dt$$

Moreover,

$$\begin{aligned} G_\alpha(z) &\sim c_0 z^{-\alpha} (1 + O(z^{1-\alpha})) & z \rightarrow 0 \\ G_\alpha(z) &\sim c_\infty z^{-1/2} e^{-\delta z} (1 + O(z^{-1})) & z \rightarrow \infty \\ \text{with } c_\infty &= (2\pi\beta)^{-1/2} \alpha^{\beta/2}. & c_0 = (2\pi\beta)^{-1} \Gamma(\alpha+1) \sin \pi\alpha \end{aligned}$$

The following estimate of  $G_\alpha(x)$  on the real line is an immediate consequence of this representation.

**Corollary 4.2.** *Let*

$$\begin{aligned} A_{\pm} &= \sup_{x \in [0,1]} [x^{\alpha} e^{\delta x} G_{\alpha}(x)]^{\pm 1} \\ B_{\pm} &= \sup_{x \in ]1,+\infty[} [x^{1/2} e^{\delta x} G_{\alpha}(x)]^{\pm 1} \end{aligned}$$

*then  $A_+, B_+, A_-, B_-$  are finite and non zero. Moreover, let*

$$f_1(x) = x^{-\alpha} e^{-\delta x} 1_{[0,1]}(x), \quad f_2(x) = x^{-1/2} e^{-\delta x} 1_{]1,+\infty[}$$

*then*

$$A_- f_1(x) + B_- f_2(x) \leq G_{\alpha}(x) \leq A_+ f_1(x) + B_+ f_2(x)$$

For the proof of Theorem 4.1 we need first to study the behavior of  $G_{\alpha}$  near the boundary  $] -\infty, 0[$ . Using (2.4) one gets

$$(4.2) \quad G_{\alpha}(-r^+) = (2i\pi\beta)^{-1} \int_0^{\infty} e^{-rt} (e^{rt^{\alpha}} - e^{rt^{\alpha} e^{-2i\pi\alpha}}) dt$$

**Proposition 4.3.** *For  $r > 0$  one has*

- (1)  $\Im(G_{\alpha}(-r^+)) < 0$ .
- (2)  $G_{\alpha}(-r^+) = c_0 r^{-\alpha} e^{-i\pi\alpha} (1 + O(r^{1-\alpha}))$  for  $r \rightarrow 0$
- (3)  $G_{\alpha}(-r^+) = -ic_{\infty} r^{-1/2} e^{\delta r} (1 + O(r^{-1}))$  for  $r \rightarrow \infty$ .

*Proof.* -(1) follows from

$$\Re(e^{rt^{\alpha}} - e^{rt^{\alpha} e^{-2i\pi\alpha}}) = e^{rt^{\alpha}} - e^{rt^{\alpha} \cos 2\pi\alpha} \cos[rt^{\alpha} \sin(2\pi\alpha)] \geq 0$$

-(2) The change of variables  $t \rightarrow \frac{t}{r}$  in (4.2) gives

$$G_{\alpha}(-r^+) = (2i\pi\beta r)^{-1} \int_0^{+\infty} e^{r^{1-\alpha} t^{\alpha}} - e^{r^{1-\alpha} t^{\alpha} e^{-2i\pi\alpha}} e^{-t} dt$$

The function

$$E(z) = (2i\pi\beta z)^{-1} \int_0^{+\infty} e^{zt^{\alpha}} - e^{zt^{\alpha} e^{-2i\pi\alpha}} e^{-t} dt$$

is entire and  $E(0) = (\pi\beta)^{-1} e^{-i\pi\alpha} \Gamma(\alpha + 1) \sin \pi\alpha$ , moreover one has  $G_{\alpha}(-r^+) = r^{-\alpha} E(r^{1-\alpha})$  from which (2) follows.

-(3) Using Laplace method we obtain the following estimate

$$(2i\pi\beta)^{-1} \int_0^{\infty} e^{-r(t-t^{\alpha})} dt = -ic_{\infty} r^{-1/2} e^{\delta r} (1 + O(r^{-1})) \quad r \rightarrow \infty$$

Moreover,

$$|e^{-\delta r} G_{\alpha}(-r^+) - (2i\pi\beta)^{-1} \int_0^{\infty} e^{-r(t-t^{\alpha}+\delta)} dt| \leq (2\pi\beta)^{-1} \int_0^{\infty} e^{-r(t-t^{\alpha}+\delta)} e^{-(1-\cos 2\pi\alpha)t^{\alpha}r} dt.$$

Since  $e^{-r(t-t^{\alpha}+\delta)} \leq 1$ , this integral is bounded above by  $(1-\cos 2\pi\alpha)^{-1/\alpha} \Gamma(1/\alpha) r^{-1/\alpha}$

□

*Proof of Theorem 4.1.*



Let  $G_\alpha(-r^+) = R(r)e^{-i\pi\theta(r)}$  be the polar decomposition of  $G_\alpha(-r^+)$ . Since  $\Im G_\alpha(-r^+)$  is negative, we can choose  $\theta(r) \in ]0, 1[$  and continuous. Proposition 4.3 implies that

$$\begin{aligned}\theta(r) &= \alpha + O(r^{1-\alpha}) & r \rightarrow 0 \\ \theta(r) &= 1/2 + O(1/r) & r \rightarrow +\infty\end{aligned}$$

Let

$$L_\alpha(z) = \exp \int_0^\infty \left[ \frac{1}{z+t} - \frac{1}{1+t} \right] \theta(t) dt,$$

this function is analytic on  $\mathbf{C} \setminus ]-\infty, 0]$  and satisfies, by well known properties of Stieltjes transforms,

$$\frac{L_\alpha(-r^+)}{L_\alpha(-r^-)} = e^{-2i\pi\theta(r)} \quad r > 0,$$

furthermore, since  $\theta(t) = 1/2 + O(1/t)$ , the integral  $\int_0^\infty \frac{1}{t+1}(\theta(t) - 1/2)dt$  is finite and  $z^{1/2}L_\alpha(z) = \exp \int_0^\infty \left[ \frac{1}{z+t} - \frac{1}{1+t} \right] (\theta(t) - 1/2)dt$  therefore

$$z^{1/2}L_\alpha(z) \rightarrow_{z \rightarrow \infty} \exp \int_0^\infty \frac{1}{1+t}(\theta(t) - 1/2)dt = C > 0$$

and  $L_\alpha(z) \sim Cz^{-1/2}$ ,  $z \rightarrow \infty$ . A similar argument, using the fact that  $\theta(t) = \alpha + O(t^{1-\alpha})$   $t \rightarrow 0$ , gives

$$z^\alpha L_\alpha(z) \rightarrow_{z \rightarrow 0} \exp \int_0^\infty \frac{1}{t(1+t)}(\theta(t) - \alpha)dt = D > 0$$

On the other hand,

$$\frac{G_\alpha(-r^+)}{G_\alpha(-r^-)} = e^{-2i\pi\theta(r)}$$

therefore the function  $E_\alpha(z) = \frac{e^{\delta z} G_\alpha(z)}{L_\alpha(z)}$  is analytic on  $\mathbf{C} \setminus ]-\infty, 0]$ , and has a continuous extension to  $\mathbf{C} \setminus \{0\}$ . It is also continuous at 0, because both  $L_\alpha(z)$  and  $G_\alpha(z)$  are equivalent to  $z^{-\alpha}$  up to a multiplicative constant, for  $z \rightarrow 0$ . By Morera's theorem, the function  $E_\alpha$  can be extended to an entire function. Moreover, since the two functions  $e^{\delta z} G_\alpha(z)$  and  $L_\alpha(z)$  are equivalent to  $z^{-1/2}$  at infinity up to a multiplicative constant,  $E_\alpha(z)$  is bounded on  $\mathbf{C}$ . Finally, by Liouville theorem,  $E_\alpha$  is constant and this constant, equal to  $e^\delta G_\alpha(1)$  is positive.  $\square$

**Remark :** If  $H$  is an HCM function then  $\frac{\log H(x)}{x}$  is bounded, thus  $G_\alpha(x^h)$  is not HCM for any  $h > 0$ . Consequently, if  $g_\alpha(x^\gamma)$  is HCM then  $\gamma < \beta^{-1}$ .

## 5. HCM, NON HCM, ANTI HCM PROPERTY OF $G_\alpha$

**Theorem 5.1.** (1) For  $\alpha \in ]1/2, 1]$ , the function  $\theta$  is decreasing and  $G_\alpha$  is anti-HCM,  
(2) For  $\alpha \in [1/3, 1/2]$ , the function  $\theta$  is increasing and  $G_\alpha$  is HCM,  
(3) For  $\alpha \in ]0, 1/3[$ , the function  $\theta$  is not monotonous and  $G_\alpha$  is neither HCM, neither anti-HCM.

For the proof we need some preliminary results.

**Lemma 5.2.** (1)  $r^\alpha \Im(G_\alpha(-r^+))$  is negative and decreasing  
(2) For  $\alpha \in ]1/3, 1]$ ,  $\text{sign}(1/2 - \alpha)r^\alpha \Re(G_\alpha(-r^+))$  is positive and increasing  
(3) For  $\alpha \leq 1/3$ ,  $r^{\alpha-1} \Re(G_\alpha(-r^+))$  is not of constant sign.

*Proof.* From (4.2) we get:

$$\Im G_\alpha(-r^+) = -(2\pi\beta)^{-1} \int_0^\infty e^{-rt} \Re(e^{rt^\alpha} - e^{rt^\alpha e^{-2i\pi\alpha}}) dt$$

$$\Re(G_\alpha(-r^+)) = (2\pi\beta)^{-1} \int_0^\infty e^{-rt} \Im(e^{rt^\alpha} - e^{rt^\alpha e^{-2i\pi\alpha}}) dt$$

The change of variables  $t \rightarrow t/r$  in the first identity gives

$$\Im(G_\alpha(-r^+)) = (2\pi\beta r)^{-1} \int_0^\infty \Re(e^{t^\alpha r^{1-\alpha}} - e^{t^\alpha r^{1-\alpha} e^{-2i\pi\alpha}}) e^{-t} dt$$

while  $t \rightarrow t/r^\alpha$  in the second gives

$$\Re(G_\alpha(-r^+)) = (2\pi\beta r^\alpha)^{-1} \int_0^\infty \Im(e^{-r^{\frac{1-\alpha}{\alpha}} t} e^{i\pi(1-2\alpha)t^\alpha}) dt$$

The function

$$r^{-1+\alpha} \Re(e^{t^\alpha r^{1-\alpha}} - e^{t^\alpha r^{1-\alpha} e^{-2i\pi\alpha}}) = \sum_n \frac{t^{n\alpha} r^{(1-\alpha)(n-1)}}{n!} (1 - \cos 2n\pi\alpha)$$

is increasing in  $r$  for all  $t > 0$ . It follows that  $r^\alpha \Im(G_\alpha(-r^+))$  is increasing.

The second identity and (2.3) give

$$r^\alpha \Re(G_\alpha(-r^+)) = \beta^{-1} x^{\alpha+1} g_{\alpha,1-2\alpha}(x)$$

( for  $x = r^{-\frac{1}{\beta}}$  )

The end of the lemma follows from 2.9. □

*Proof of theorem 5.1 :* According to section 2.1 it is enough to consider monotonicity properties of  $\theta$ . Recall that, for all  $\alpha$ ,  $\theta(0) = \alpha$  and  $\theta(+\infty) = 1/2$ , moreover  $\Im(G_\alpha(-r^+)) = R(r) \sin \pi\theta(r)$  is negative and decreasing and, for  $\alpha \geq 1/3$ ,  $\Re(G_\alpha(-r^+)) = R(r) \cos \pi\theta(r)$  has constant sign and is monotonous. It follows that  $G_\alpha(-r^+)$  takes all its values in a quarter plane and  $\theta(r)$  has a constant sign and its absolute value is increasing, thus  $\theta$  is monotonous, decreasing for  $\alpha > 1/2$  and increasing for  $\alpha \in [1/3, 1/2]$ .

Finally for  $\alpha \in ]0, 1/3[$ , we obtain that  $\Re(G_\alpha(-r^+)) = R(r) \cos \pi\theta(r)$  can take negative values, thus  $\theta(r)$  does not take all its value inside the interval  $[\alpha, 1/2]$ , thus it is not monotonous.

In the case  $\alpha \in [1/3, 1[$  we obtain a better estimate for  $G_\alpha(x)$  than in corollary 4.2.

**Corollary 5.3.** *Let*

$$f_1(x) = x^{-\alpha} e^{-\delta x} 1_{[0,1]}(x), \quad f_2(x) = x^{-1/2} e^{-\delta x} 1_{]1,+\infty[}$$

*If  $\alpha \in [1/3, 1/2]$ , then*

$$G_\alpha(1)f_1(x) + c_\infty f_2(x) \leq G_\alpha(x) \leq c_0 f_1(x) + G_\alpha(1)f_2(x)$$

*If  $\alpha \in ]1/2, 1]$ , then*

$$G_\alpha(1)f_1(x) + c_\infty f_2(x) \geq G_\alpha(x) \geq c_0 f_1(x) + G_\alpha(1)f_2(x)$$

Using the proposition 2.5, we also obtain new GGC densities related to  $\alpha$ -stable densities.

$[\varepsilon(\alpha - 1/2)]$  denotes the sign of  $\alpha - 1/2$ .

**Corollary 5.4.** *If  $\alpha \in [1/3, 1]$  then*

- 1) *The function  $[c_0^{-1}x^{-\alpha}G_\alpha(x)]^{-\varepsilon(\alpha-1/2)}$  is the Laplace transform of a random variable of the form  $Y - \varepsilon(\alpha - 1/2)\delta$  where  $Y$  is GGC.*
- 2)  *$[c_\infty^{-1}x^{-1/2}e^{\delta/x}G_\alpha(1/x)]^{\varepsilon(\alpha-1/2)}$  is the Laplace transform of a GGC random variable.*

Finally we obtain another consequence for the  $\alpha$ -densities.

**Corollary 5.5.** *If  $\alpha > 1/2$  then  $S_\alpha$  is GGC and  $S_\alpha^{-\beta}$  is not GGC.*

*proof* The GGC property of  $S_\alpha$  is known and has already been already been mentioned in paragraph 2.2. Consider the Laplace transform, for  $\lambda \geq 0$ ,

$$Lp(S_\alpha^{-\beta})(\lambda) = \int_0^\infty e^{-\lambda x} G_\alpha(x) dx = \int_0^\infty e^{-t} G_\alpha\left(\frac{t}{\lambda}\right) \frac{dt}{\lambda}$$

Since  $(z^{1/2}1_{|z|>1} + z^\alpha 1_{|z|\leq 1})e^{\delta z}G_\alpha(z)$  is a bounded function of  $z$ , the integral can be analytically continued by an analytic to  $\mathbf{C} \setminus ]-\infty, 0] \cap \{|z| > \delta\}$  and this continuation satisfies again for  $r > \delta$ ,

$$Lp(S_\alpha^{-\beta})(-r^+) = - \int_0^\infty e^{-t} G_\alpha\left(-\left(\frac{t}{r}\right)^-\right) \frac{dt}{r} = - \int_0^\infty e^{-rt} G_\alpha(-t^-) dt$$

Since  $\Im(G_\alpha(-t^-))$  is positive and increasing while  $\Re(\Im(G_\alpha(-t^-)))$  is negative and increasing, the same is true for  $-\Im Lp(S_\alpha^{-\beta})(-r^+)$  and  $-\Re(Lp(S_\alpha^{-\beta})(-r^+))$ , consequently the opposite of the argument of  $Lp(S_\alpha^{-\beta})(-r^+)$  is decreasing again for  $r > \delta$ .

Thus  $Lp(S_\alpha^{-\beta})(x)$  cannot be HCM, consequently  $S_\alpha^{-\beta}$  is not GGC.

## 6. SOME FURTHER PROPERTIES OF GGC AND HCM FUNCTIONS

**Theorem 6.1.** *Let  $H$  be an HCM function and  $g$  be a GGC density, then the function  $\int_0^\infty H(xy)g(y)dy$  is HCM if it is finite.*

For the proof of this result we derive some lemmas. The first one is due to Bondesson [2].

**Lemma 6.2.** *The product and the ratio of two independent GGC random variables is GGC.*

From this we deduce:

**Lemma 6.3.** *Let  $g$  be a GGC density and  $\beta$  a real number such that  $m_\beta = \int_0^\infty x^\beta g(x)dx$  is finite, then  $m_\beta^{-1}x^\beta g(x)$  is a GGC density.*

*Proof.* Let  $H$  be the Laplace transform of a GGC random variable  $Y$ , and  $X$  be a GGC random variable with density  $g$  independent of  $Y$ .

The function  $\int_0^\infty H(xy)g(y)dy$  is the Laplace transform of  $XY$ . According to lemma 6.2 the independent product  $XY$  is GGC again, thus  $\int_0^\infty H(xy)g(y)dy$  is the Laplace transform of a GGC variable. Thus it is HCM. Replacing  $H(x)$  by this

to  $H(x)e^{-x}(\varepsilon^{-\beta}(\varepsilon+x)^{-\beta})$  which is the Laplace transform of  $Y + E_\varepsilon + 1$  where  $E_\varepsilon$  has  $\Gamma(\varepsilon, \beta)$ -distribution, we obtain that the integral

$$\int_0^\infty H(xy)\varepsilon^{-\beta}(\varepsilon+xy)^{-\beta}g(y)dy$$

is HCM. Multiply this integral by the constant  $\varepsilon^\beta$  and let  $\varepsilon \rightarrow 0$ , the monotone convergence theorem, and the fact that HCM property is stable by multiplication by a positive constant and by pointwise limit implies that

$$\int_0^\infty e^{-xy}(xy)^{-\beta}g(y)dy$$

is HCM. Mutiplying this integral by  $m_\beta^{-1}x^\beta$  we again get an HCM function and the integral obtained is the Laplace transform of the density  $m_\beta^{-1}x^{-\beta}g(x)$ . Since this Laplace transform is HCM the density is GGC.  $\square$

**Lemma 6.4.** *Let  $\theta$  be an increasing function and  $H$  the associated HCM funtion*

$$H(x) = \exp \int_0^\infty \left( \frac{1}{x+t} - \frac{1}{1+t} \right) \theta(t) dt$$

*Then  $H$  is a pointwise limit of HCM functions  $H_n$  whose  $\theta$ -function in the representation (2.2) is bounded. Moreover one can chose the  $H_n$  such that for all  $\epsilon > 0$  there exists  $N$  s.t. if  $n > N$  and  $x \in ]0, +\infty[$  then*

$$(1 - \epsilon)H(x) \leq H_n(x) \leq H(x)e^{\epsilon(x+x^{-1})}$$

*Proof.* Let  $n$  be a positive integer and

$$\theta(t) = \theta_n + (\theta(t) - n)1_{\theta(t) \geq n} + (\theta(t) + n)1_{\theta(t) \leq -n}$$

with

$$\theta_n = \theta(t) \vee (-n) \wedge n$$

Moreover let

$$\begin{aligned} H_n(x) &= \exp \int_0^\infty \left( \frac{1}{x+t} - \frac{1}{1+t} \right) \theta_n(t) dt \\ E_n(x) &= \int_0^\infty \left( \frac{1}{x+t} - \frac{1}{1+t} \right) (\theta(t) - n) 1_{\theta(t) > n} dt \\ \hat{E}_n(x) &= \int_0^\infty \left( \frac{1}{x+t} - \frac{1}{1+t} \right) (\theta(t) + n) 1_{\theta(t) < -n} dt \end{aligned}$$

Clearly

$$H = \hat{E}_n H_n E_n$$

Since  $\frac{1}{x+t} - \frac{1}{1+t}$  and  $x - 1$  have the same sign and

$$-1_{x>1} \inf(x, t^{-2}) \leq \frac{1}{x+t} - \frac{1}{1+t} \leq 1_{x<1} \inf(x^{-1}, t^{-2})$$

we obtain

$$e^{-\varepsilon_n x 1_{x>1}} \leq E_n \leq e^{\varepsilon_n 1_{x<1}}$$

and

$$e^{-\hat{\varepsilon}_n x^{-1} 1_{x<1}} \leq \hat{E}_n \leq e^{\hat{\varepsilon}_n 1_{x>1}}$$

with

$$\varepsilon_n = \int_0^\infty 1_{\theta(t) > n} (\theta(t) - n) \frac{dt}{t^2}$$

and

$$\hat{\varepsilon}_n = \int_0^\infty 1_{\theta(t) < -n} (-n - \theta(t)) dt$$

The positive numbers  $\varepsilon_n$  and  $\hat{\varepsilon}_n$  go to zero when  $n \rightarrow +\infty$  for all  $\varepsilon > 0$ , let  $N$  such that  $\varepsilon > \varepsilon_n \vee \hat{\varepsilon}_n$  then  $H_n(x)$  satisfies the required estimate of the lemma.  $\square$

*Proof of Theorem 6.1*

Let  $X$  and  $Y$  be GGC random variables, let  $H$  be the Laplace transform of  $Y$  and  $g$  be the density of  $X$ . On the other hand, the sequence of functions  $(1 + \frac{b}{xn})^{-n} = x^n \frac{b}{n} (1 + \frac{nx}{b})^{-n}$  have limit  $e^{-\frac{b}{x}}$  and they are bounded by 1. Moreover the function  $(1 + \frac{xn}{b})^{-n}$  is the Laplace transform of a  $\Gamma(n, \frac{n}{b})$ - random variable  $E_n$ , The product  $H(x)(1 + \frac{nx}{b})^{-n}$  is the Laplace transform of the GGC variable  $Y + E_n$ , since the independant product  $X(Y + E_n)$  is again GGC and its Laplace transform is

$$\int_0^\infty H(xy) (1 + \frac{nx}{b})^{-n} g(y) dy$$

Thus, this function is HCM.

Suppose that the random  $X$  has moments of all order, according to lemma 6.3 the function  $g(y)$  can be replaced by  $g(y)y^{\beta+n}$  for any  $n$  and  $\beta$ , and again the integral

$$\int_0^\infty H(xy) (1 + \frac{nx}{b})^{-n} y^{\beta+n} g(y) dy$$

defines an HCM function.

Multiply by  $x^n (\frac{b}{n})^n$  and let  $n$  goes to infinity and use the dominated convergence theorem in order to obtain that for all real  $\beta$  and  $b > 0$ .

$$\int_0^\infty H(xy) e^{-\frac{b}{xy}} y^\beta g(y) dy$$

is HCM.

The function  $H(x)e^{-ax}$  can replace  $H(x)$  in this formula, since it is also a Laplace transform of  $Y + a$  which is a GGC variable again and

$$\int_0^\infty H(xy) e^{-axy} . e^{-\frac{b}{xy}} y^\beta g(y) dy$$

is also HCM.

Take  $a > 0$  and  $b > 0$ , the hypothesis that  $X$  (with density  $g$ ) has moments of all orders can be removed because the GGC densities with finite moments are dense in the family of GGC densities for the weak topology. Finally, the function

$$\int_0^\infty H(xy) e^{-axy} . e^{-\frac{b}{xy}} y^\beta g(y) dy$$

is HCM for any  $H$  which is the Laplace transform of a GGC density, any real  $\beta$  and any GGC-density  $g$ .

Let  $H$  be any HCM function of the form

$$H(x) = \exp \int_0^\infty (\frac{1}{x+t} - \frac{1}{1+t}) \theta(t) dt$$

for an increasing function  $\theta$ , and  $(H_n)$  be a sequence of HCM functions approaching  $H$  as it is described in lemma 6.4. The  $\theta$  functions of  $H_n$  are bounded below (say by  $-n$ ), then  $H_n$  are of the form  $x^n \tilde{H}_n(x)$  where  $\tilde{H}_n$  are Laplace transform of GGC-variables. (see proposition 2.5). Thus, the functions

$$\int_0^\infty H_n(xy) e^{-axy} e^{\frac{-b}{xy}} (xy)^n y^\beta g(y) dy$$

are HCM.

Divide by  $x^{-n}$  and let  $\beta = -n$  and obtain that the functions

$$\int_0^\infty H_n(xy) e^{-axy} e^{\frac{-b}{xy}} g(y) dy$$

are HCM.

Finally, by Lebesgue dominated convergence in  $L^1(R^+, e^{-\varepsilon(x+x^{-1})})$  with  $0 < \varepsilon < \inf(a, b)$ , the integral  $\int_0^\infty H_n(xy) e^{-axy} e^{\frac{-b}{xy}} g(y) dy$  converges for all  $x > 0$  to  $\int_0^\infty H(xy) e^{-axy} e^{\frac{-b}{xy}} g(y) dy$  and this function is again HCM.

The monotone convergence theorem enable to extend the property in case  $a$  or  $b$  are zero and the proof is finished.  $\square$

**Corollary 6.5** (T-Simon, P.Bosch). *The  $\alpha$  stable positive density,  $g_\alpha$ , is HCM if and only if  $\alpha \in ]0, 1/2]$ .*

*Proof.* The scaling property gives the identity for  $\alpha \leq 1/2$

$$S_\alpha = S_{1/2}^{\frac{1}{2\alpha}} \cdot S_{2\alpha}.$$

The density of the standard  $1/2$ -stable r.v.,  $S_{1/2}^{\frac{1}{2\alpha}}$  is  $H(x) = (2\pi)^{-1/2} \cdot \alpha x^{-3\alpha-1} e^{-x^{-2\alpha}}$  which is clearly HCM.

Applying theorem 6.1 to the HCM function  $x^{-1}H(x^{-1})$  and to the density of the GGC variable  $S_{2\alpha}$  gives the required property.  $\square$

Finally, let  $\gamma$  be the bigger power such that  $g_\alpha(x^\gamma)$  is HCM, we have obtained that  $\gamma = \beta^{-1}$  for  $\alpha \in [1/3, 1/2]$  and  $\gamma \in [1, \beta^{-1}[$  for  $\alpha < 1/3$ .

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